Higher order linear ODE: General theory

I. homogeneous case: $y^{(n)}+p_1(t)y^{(n-1)}+\cdots+p_{n-1}(t)y'+p_n(t)y=0$

* Fundamental set of solutions: $y_1(t),\cdots,y_n(t)$ function that are ① linearly independent $(W(y_1,\cdots,y_n)\neq 0)$

② solutions to the homogeneous ODE.

* Principle of superposition: If $\{y_1,\cdots,y_n\}$ forms a fundamental set of solutions, then the general solution of homogeneous ODE is $y=C_1y_1+C_2y_2+\cdots+C_ny_n$.

II. Nonhomogeneous case: $y^{(n)}+p_1(t)y^{(n-1)}+\cdots+p_{n-1}(t)y'+p_n(t)y=g(t)$

* Structure of general solution: $y=y_c+Y$

where y_c, complementary solution, is the general solution to the homog. ODE

Y, particular solution, is a solution to the nonhomogeneous ODE.

* Principle of superposition:

Y_1 sol'n to $y^{(n)}+p_1(t)y^{(n-1)}+\cdots+p_{n-1}(t)y'+p_n(t)y=g_1(t)$

Y_2 sol'n to $y^{(n)}+p_1(t)y^{(n-1)}+\cdots+p_{n-1}(t)y'+p_n(t)y=g_2(t)$

then Y_1+Y_2 is a sol'n to

$$y^{(n)}+p_1(t)y^{(n-1)}+\cdots+p_{n-1}(t)y'+p_n(t)y=g_1(t)+g_2(t).$$

Higher order linear ODE w/ constant coefficients

I. Homogeneous case.

$$a_n y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1y'+a_0y=0$$

Try $y=e^{rt} \Rightarrow y'=re^{rt}, \ldots, y^{(n)}=r^n e^{rt}$.

$$a_n r^n \cancel{e^{rt}}+a_{n-1}r^{n-1}\cancel{e^{rt}}+\cdots+a_1r\cancel{e^{rt}}+a_0\cancel{e^{rt}}=0$$

Characteristic Equation:

$$a_n r^n+a_{n-1}r^{n-1}+\cdots+a_1r+a_0=0$$

Theorem: Every real polynomial can be factorized into product of linear and quadratic polynomials.

Corollary: If $\alpha+i\beta$ is a root repeated m times, then $\alpha-i\beta$ is also a root repeated m time.

Let r is a characteristic root.

① If r is a real single root, then it contribute one function to fund. set. of sol'ns, namely e^{rt}.

② If r is a real root repeated m times, then it contributes m functions to fund. set of sol'ns:

$$e^{rt}, te^{rt}, t^2e^{rt}, \dots, t^{m-1}e^{rt}$$

③ If $r=\alpha+i\beta$ is a complex single root, then together with its conjugate $\bar{r}=\alpha-i\beta$, they contribute two functions to the fund. set of sol'n

$$e^{\alpha t}\cos\beta t, \quad e^{\alpha t}\sin\beta t$$

④ If $r=\alpha+i\beta$ is a complex root repeated m times, then $\bar{r}=\alpha-i\beta$ is also repeated m times. They contribute $2m$ functions to the fund. set of sol'ns

$$e^{\alpha t}\cos\beta t, \ e^{\alpha t}\sin\beta t, \ te^{\alpha t}\cos\beta t, \ te^{\alpha t}\sin\beta t,$$

$$\dots \ \dots \ \dots \ \dots, \ t^{m-1}e^{\alpha t}\cos\beta t, \ t^{m-1}e^{\alpha t}\sin\beta t.$$

Example: $y''' - y' = 0$

Char. eqn. $r^3 - r = 0 \Rightarrow r(r^2-1)=0 \Rightarrow r(r-1)(r+1)=0$

Char. roots: $r = 1, -1, 0.$

Gen. sol'n: $y = C_1e^{t} + C_2e^{-t} + C_3$

Example: $y''' - y'' - y' + y = 0$

Char. eqn. $r^3 - r^2 - r + 1 = 0 \Rightarrow r^2(r-1) - (r-1) = 0$

$\Rightarrow (r-1)(r^2-1) = 0 \Rightarrow (r-1)^2(r+1) = 0$

Char. roots : $r = 1, 1, -1$

Gen. sol'n $y = C_1 e^{-t} + C_2 e^t + C_3 t e^t$.

Example: $y''' - 3y'' + 3y' - y = 0$

Char. eqn. $r^3 - 3r^2 + 3r - 1 = 0$.

$r^3 - 1 - 3r^2 + 3r = 0$

$(r-1)(r^2+r+1) - 3r(r-1) = 0$

$(r-1)(r^2+r+1-3r) = 0$.

$(r-1)(r^2-2r+1) = 0 \Rightarrow (r-1)^3 = 0$

$r = 1, 1, 1$

Gen. sol'n: $y = C_1 e^t + C_2 t e^t + C_3 t^2 e^t$.

$a^2 - b^2 = (a-b)(a+b)$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$a^4 - b^4 = (a-b)(a^3 + a^2 b + a b^2 + b^3)$

Ex. $a^5 - b^5 = ?$

Example: $y''' - y = 0$.

Char. eqn. $r^3 - 1 = 0 \Rightarrow (r-1)(r^2+r+1) = 0$

Char. roots: $r = 1, \frac{-1 \pm \sqrt{1^2 - 4\cdot1\cdot1}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

Gen. sol'n: $y = C_1 e^t + C_2 e^{-\frac{1}{2}t} \cos\frac{\sqrt{3}}{2}t + C_3 e^{-\frac{1}{2}t} \sin\frac{\sqrt{3}}{2}t$

Example: $y^{(4)} + 8y'' + 16y = 0$

$r^2+4=0 \Rightarrow r^2=-4$

Char. eqn: $r^4 + 8r^2 + 16 = 0 \Rightarrow (r^2+4)(r^2+4) = 0$

Char. roots: $r = 2i, 2i, -2i, -2i$.

Gen. sol'n: $y = C_1 \cos 2t + C_2 \sin 2t + C_3 t \cos 2t + C_4 t \sin 2t$.

II. Nonhomogeneous Case.

Variation of Parameters would be too complicated.

We use method of undetermined coefficients.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = g(t)$$

Only need to know how to deal with such $g(t)$

$$g(t) = e^{\alpha t}\cos\beta t\,(a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0)$$
$$+ e^{\alpha t}\sin\beta t\,(b_n t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0)$$

or sums of such $g(t)$ (with different exp. coeff's).

Recall: First try template

$$Y(t) = (A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0)e^{\alpha t}\sin\beta t$$
$$+ (B_n t^n + B_{n-1}t^{n-1} + \cdots + B_1 t + B_0)e^{\alpha t}\cos\beta t.$$

Recall: If exp. coeff. of $g(t)$ appears as a root repeated m times, then first m tries fail.

Examples: Determine the final template for the following ODEs

(a). $y''' - y'' - y' + y = 2e^{-t}$

char. eqn: $r^3 - r^2 - r + 1 = 0 \Rightarrow (r-1)^2(r+1) = 0 \Rightarrow r = 1, 1, -1.$

Exp. coeff. $= -1$ appears as a single root.

(First try template Ae^{-t}, fails once)

$Y = Ate^{-t}.$

(b) $y''' - y'' - y' + y = 2e^{t}$

char. roots $r = 1, 1, -1.$

Exp. coeff. $= 1$ appears twice

$Y = At^2e^{t}.$

(c) $y^{(4)} + 2y^{(3)} + 2y'' = 16$

char. eqn.: $r^4 + 2r^3 + 2r^2 = 0$

$\Rightarrow r^2(r^2 + 2r + 2) = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4\times 2}}{2} = -1 \pm i, \ 0, 0.$$

Exp. coeff. $= 0$, appears twice

$Y = At^2$

(d). $y^{(4)} - y = \cos t$

$i = \sqrt{-1}$

$r^2 + 1 = 0 \Rightarrow r^2 = -1$
$\Rightarrow r = \pm\sqrt{-1}$

Char. eqn.: $\underbrace{r^4}_{(r^2)^2} - 1 = 0 \Rightarrow (r^2+1)(r^2-1) = (r^2+1)(r+1)(r-1)$

Char. roots: $1, -1, i, -i$

Exp. coeff. $= i$ appears once.

$Y = t(A\cos t + B\sin t) = At\cos t + Bt\sin t.$

(e) $y^{(4)} + 2y'' + y = t\sin t$

Char. eqn. $r^4 + 2r^2 + 1 = 0 \Rightarrow (r^2+1)^2 = 0$

Char roots: $\pm i, \pm i$ or $i, i, -i, -i$

Exp. coeff. $= i$ appears twice

(First try template $(At+B)\sin t + (Ct+D)\cos t$)

$Y = (At^3 + Bt^2)\sin t + (Ct^3 + Dt^2)\cos t$

(f) $y^{(4)} + 2y''' + 2y'' = e^{-t}\cos t$

Char. roots $r = 0, 0, -1 \pm i$

Exp. coeff. $= -1+i$ appears once

(First try template: $Ae^{-t}\cos t + Be^{-t}\sin t$)

$Y = Ate^{-t}\cos t + Bte^{-t}\sin t$

Example: $y''' - y'' - y' + y = 8e^t + 4e^{-t} + 25\cos 2t + t^2$

Char. eqn. $r^3 - r^2 - r + 1 = 0 \Rightarrow (r-1)^2(r+1) = 0$

Char. roots $r = 1, 1, -1$

Comp. sol'n: $y_c = C_1 e^t + C_2 te^t + C_3 e^{-t}$.

Set Y_1 sol'n to $y''' - y'' - y' + y = 8e^t$

Exp. coeff $= 1$, appears twice.

$$Y_1 = At^2 e^t$$

$$Y_1' = 2Ate^t + At^2e^t$$

$$Y_1'' = 2A \cdot e^t + 2A \cdot te^t + 2Ate^t + At^2e^t$$

$$= 2Ae^t + 4Ate^t + At^2e^t$$

$$Y_1''' = 2Ae^t + 4A(e^t + te^t) + 2Ate^t + At^2e^t$$

$$= 6Ae^t + 6Ate^t + At^2e^t$$

$$Y_1''' - Y_1'' - Y_1' + Y_1 = 6Ae^t + 6Ate^t + At^2e^t$$

$$- 2Ae^t - 4Ate^t - At^2e^t - 2Ate^t - At^2e^t + At^2e^t$$

$$= 6Ae^t - 2Ae^t = 4Ae^t = 8e^t \Rightarrow A = 2$$

$$Y_1 = 2t^2e^t.$$

Set Y_2 sol'n to $y''' - y'' - y' + y = 4e^{-t}$.

Exp. coeff $= -1$. appears once.

$Y_2 = Ate^{-t}$.

$Y_2' = Ae^{-t} - Ate^{-t}$.

$Y_2'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$

$Y_2''' = 2Ae^{-t} + Ae^{-t} - Ate^{-t} = 3Ae^{-t} - Ate^{-t}$

$$
\begin{aligned}
Y_2''' - Y_2'' - Y_2' + Y_2 &= 3Ae^{-t} - Ate^{-t} + 2Ae^{-t} - Ate^{-t} \\
&\quad - Ae^{-t} + Ate^{-t} + Ate^{-t} \\
&= 4Ae^{-t} = 4e^{-t} \Rightarrow A = 1
\end{aligned}
$$

$Y_2 = te^{-t}$.

Set Y_3 sol'n to $y''' - y'' - y' + y = 25\cos 2t$

Exp. coeff $= 2i$ does not appear as a root

$Y_3 = A\cos 2t + B\sin 2t$

$Y_3' = -2A\sin 2t + 2B\cos 2t$

$Y_3'' = -4A\cos 2t - 4B\sin 2t$

$Y_3''' = +8A\sin 2t - 8B\cos 2t$

$Y_3''' - Y_3'' - Y_3' + Y_3 = 8A\sin 2t - 8B\cos 2t + 4A\cos 2t + 4B\sin 2t$

$$+2A\sin 2t - 2B\cos 2t + A\cos 2t + B\sin 2t$$

$$= (10A+5B)\sin 2t + (5A-10B)\cos 2t$$

$$= 25\cos 2t$$

$\Rightarrow 10A+5B=0, \quad 5A-10B=25$

$\Rightarrow B=-2A, \Rightarrow 5A-10(-2A)=25A=25 \Rightarrow A=1, B=-2$

$Y_3 = \cos 2t - 2\sin 2t$

Set Y_4 sol'n of $y''' - y'' - y' + y = t^2$

Exp. coeff $=0$, not a root.

$Y_4 = At^2 + Bt + C$

$Y_4' = 2At+B, \quad Y_4'' = 2A, \quad Y_4''' = 0$

$$Y_4''' - Y_4'' - Y_4' + Y_4 = 0 - 2A - 2At - B + At^2 + Bt + C$$

$$= At^2 + (B-2A)t + C - B - 2A$$

$$= t^2$$

$\Rightarrow A=1, \quad B-2A=0, \quad C-B-2A=0$

$\Rightarrow A=1, \quad B=2, \quad C=B+2A=4$

$Y_4 = t^2+2t+4.$

Gen. sol'n: $y = C_1 e^{t} + C_2 t e^{-t} + C_3 e^{-t}$

$+ 2t^2 e^{t} + te^{-t} + \cos 2t - \sin 2t + t^2 + 2t + 4.$

Lecture Notes of Differential Equation